NPS-53Fe75041

NAVAL POSTGRADUATE SCHOOL

Monterey, California

LOCALLY DETERMINED SMOOTH INTERPOLATION AT IRREGULARLY SPACED POINTS IN SEVERAL VARIABLES

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April 1975

Technical Report For Period
January 1975-March 1975

Approved for public release; distribution unlimited

Prepared for:
...f of Naval Research, Arlington, Virginia 22217

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The work reported herein was supported in part by the Foundation Research Program of the Naval Postgraduate School with funds provided by the Chief of Naval Research.

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SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER NPS-53Fe75041	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (*and Subtitle*) Locally Determined Smooth Interpolation at Irregularly Spaced Points in Several Variables		5. TYPE OF REPORT & PERIOD COVERED Technical Report January 1975-March 1975
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Richard Franke		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Foundation Research Program Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61152N; RR000-01-10; N0001475WR50001
11. CONTROLLING OFFICE NAME AND ADDRESS Chief of Naval Research Arlington, Virginia 22217		12. REPORT DATE April 1975
		13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS (*if different from Controlling Office*)		15. SECURITY CLASS. (*of this report*) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (*of this Report*)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (*of the abstract entered in Block 20, if different from Report*)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (*Continue on reverse side if necessary and identify by block number*)

Interpolation, Several Variables, Smooth Interpolation

20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

A class of methods for local interpolation at irregularly spaced points for functions of two or more variables is developed. The methods are based on a weighted average of the values of local interpolating functions, with the local interpolating functions and the weighting functions chosen so as to incorporate the desired smoothness. Numerical results for several interpolation functions from this class are compared with global approximations, some of which are local when implemented on a computer.

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S/N 0102-014-6601

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1.0 Introduction

The problem which we will be considering is that of interpolation in two (or more) variables, where the data to be interpolated is at irregularly spaced points. While the discussion will often center on functions of two variables, most of the ideas are readily extendable to any number of independent variables.

Assume that data points (X_k, f_k) , $k = 1,\ldots,N$ are specified, where the X_k lie in n-space. We make no assumption about the disposition of the points X_k . We wish to construct a function $F(X)$ which is defined in some region which contains all the X_k and such that $F(X_k) = f_k$, $k = 1,\ldots,N$. We wish to discuss a broad enough class of interpolating functions so that we can require that $F(X)$ is not only continuous, but perhaps has some continuous derivatives as well.

A number of methods have been proposed for the problem. For the most part these are global methods, most of which require the solution of a system of N linear equations for the coefficients of a set of basis functions. Typical of these are polynomial interpolation [12], and optimal approximations in various spaces of functions [6], [7], [9], [2].

Another type of global approximation (although the computational version is local) is the idea of taking the function value to be a weighted average of the values at data points, the weight being a decreasing function of distance [11]. A typical weight function is $w(d) = d^{-2}$. There are several serious faults with this approximation, such as the first partial derivative being zero at each data point. These faults are overcome by the imposition of somewhat artificial conditions on a variation of the approximation. A remaining bad feature is that the approximation is dependent on the coordinate system, or rather, the measure of distance.

A variation of the distance weighting idea is proposed by McLain [5]. There the idea is to fit the data with an equation of specified type in the weighted least squares sense. The weight attached to each point is a decreasing function of its distance from the point at which the approximation is desired. McLain investigated a number of different fitting functions and weight functions. This approach eliminates in a natural way most of the bad features of Shepard's approximation, although not the dependence on the coordinate system. McLain claims the approximation has partial derivatives of all orders, and although this seems plausible, it is neither obvious nor easily verifiable.

Where the data is on a somewhat regular set of points, e.g., if the grid is topologically equivalent to a regular grid [3], or if the grid can be triangulated, or represented as a union of convex quadrilaterals [4], special approaches may give good results. Approximations in these cases would generally be local with limited smoothness, although Ferguson's [3] parametric representation is global and a type of spline.

Another approach to the interpolation problem, the idea of which is the basis for most of our investigation, is due to Maude [8]. The resulting approximation can be made to have arbitrarily high smoothness and is locally determined. Since we will discuss the idea at length in the next section, it is mentioned here only for completeness.

The ideas which will be important to our discussion will include the following: (i) No assumption will be made about the disposition of the data points, and in particular we do not preclude the possibility of a regular grid of points; (ii) A smooth approximation in the sense that the function and some of its partial derivatives are continuous, is desirable; (iii) The number of points, N, may be so large that the approximation must be locally determined, or at least have a local basis (such as the B-splines are a local basis for univariate spline approximation) to be computationally viable; (iv) It is desirable for the approximation to be independent of translation and contraction or expansion of the coordinate system; and (v) It is desirable that the procedure be robust so that failure of the existence of the approximation can perhaps be circumvented by some slight alteration.

It is likely that (iii) rules out the optimal approximations because they are global, and the basis functions are sufficiently complex that it is quite unlikely a set of basis functions can be derived which have compact support. A large number of points also rules out polynomial approximation.

2.0 A Class of Interpolation Methods

We will begin this section with a brief review of a method due to Maude [8]. Some of the difficulties experienced with the method will be mentioned. A general class of methods which contains Maude's will be developed and some properties of the class of methods will be discussed.

2.1 Maude's Method

Let (x_k, y_k) and $f_k = f(x_k, y_k)$, $k = 1,\ldots,N$ be specified. With each point (x_n, y_n) associate a circle C_n, of radius r_n with center at (x_n, y_n). This circle is chosen so that the closed disk contains (x_n, y_n) and its five nearest neighbors from the set $\{(x_k, y_k)\}$, with no more than four of the neighbors in the interior. There may be 'ties' for the position of fifth nearest point, and one needs some means of deciding which point on the boundary is to be used in the subsequent calculations. Let the second degree polynomial interpolating the function $f(x,y)$ at (x_n,y_n) and its five nearest neighbors be denoted by $Q_n(x,y)$. Let

$$w(s) = \begin{cases} 1-3s^2 + 2s^3 , & 0 \le s \le 1 \\ 0 , & s > 1 \end{cases} .$$

To obtain the value of the interpolating function $F(x,y)$ at the point (x^*,y^*), we calculate a weighted average of the values of $Q_n(x^*,y^*)$ for those values of n such that (x^*,y^*) lies in the circle C_n . In particular,

$$F(x^*,y^*) = \sum_{n=1}^{N} w\left(\sqrt{(x^*-x_n)^2 + (y^*-y_n)^2}\Big/r_n\right)Q_n(x^*,y^*)\Big/\sum_{n=1}^{N} w\left(\sqrt{(x^*-x_n)^2 + (y^*-y_n)^2}\Big/r_n\right)$$

Because $w\left(\sqrt{(x -x)^2 + (y^*-y_n)^2}\Big/r_n\right)$ is zero unless (x^*,y^*) lies inside the circle C_n , the sum can be taken over only those values of n such that (x^*,y^*) is in C_n . The function is defined for all points (x^*,y^*) which lie in the interior of one or more of the circles. By virtue of the weights being zero, with zero first partial derivatives, on the boundary of the circles, the interpolating function F is continuous, with continuous first partial derivatives.

We can note that F is a bivariate analogue of the quintic spline of deficiency three [1]. Unfortunately, unlike the univariate case, the bivariate version above can fail. It is not necessarily true that the polynomials Q_k must exist. In fact, on a uniform grid a boundary point and its five nearest neighbors will lie on two straight lines, hence the interpolating polynomial does not generally exist. Non-existence of a particular Q_k can be bypassed by simply ignoring that circle in the definition of F . This is handled computationally by setting $r_k = 0$. For isolated instances the effect may be minimal, although failure for very many points could severely curtail the region of definition of the interpolating function and even exclude some data points.

Experience with this version of the computational scheme indicates that the behavior of the Q_k are adversely affected by the presence of even moderately steep gradients. This problem is not unique to polynomials, but they do seem to be more sensitive than other basis functions we have investigated. In addition, regions where the data points are relatively sparse seem to yield poor approximations.

2.2 A Generalization

We will develop a class of interpolation methods based on Maude's idea, which will contain his method as a special case. We will begin with a weighting function associated with each point, although in the application it will probably not be the starting point.

We presume that data points (x_k,y_k) and corresponding function values f_k , $k = 1,\dots,N$ have been given. With each point (x_n,y_n) associate a nonnegative weight function $w_n(x,y)$ which has compact support S_n , such that (x_n,y_n) is contained in the interior of S_n . Let $U_n(x,y)$ be a local interpolating function which is defined on S_n and interpolates to the value f_k at each (x_k,y_k) in S_n .

We define the value of the interpolating function $F(x,y)$ at the point (x^*,y^*) to be

$$F(x^*,y^*) = \sum_{n=1}^{N} w_n(x^*,y^*)U_n(x^*,y^*) \Big/ \sum_{n=1}^{N} w_n(x^*,y^*) \quad ,$$

provided (x^*,y^*) lies in the interior of one of the S_n . Since presumably (x^*,y^*) does not lie in all S_n , we may want to write

$$F(x^*,y^*) = \sum_{n\in I^*} w_n(x^*,y^*)U_n(x^*,y^*) / \sum_{n\in I^*} w_n(x^*,y^*)$$

where $$I^* = \{ n : (x^*,y^*) \in \text{interior } (S_n) \} .$$

We can make a number of observations about the function F .

(a) $F(x_k,y_k) = f_k$ since if $(x_k,y_k) \in$ interior of S_n , $U_n(x_k,y_k) = f_k$; (b) F is at least as smooth as the least smooth of the functions w_k and U_k , $k = 1,\ldots,N$; (c) $F(x^*,y^*)$ is locally determined by the points $(x_k,y_k) \in \bigcup_{n\in I^*} S_n$; (d) the error in the interpolation is no worse than that of the poorest interpolation by a U_k , since the value of F is a convex combination of the U_k , hence the error is a convex combination of the local interpolating errors.

It is trivially observed that Maude's method is a special case, provided the Q_k (U_k above) exist. For Maude's method the error is of the form $O(h^2)$, where $h = \max_k r_k$ [12]. If one considers a sequence of approximations as more points are added, in a fashion such that $h \to 0$, it is necessary that the 'condition' of the sets of nearest neighbors not deteriorate too much in order to maintain the $O(h^2)$ error estimate.

3.0 Implementation

In order for a scheme such as that outlined in the previous section to be reasonably easy to implement, the weight functions, in connection with the shape of the support regions S_k must be chosen with some care. Since we wish to be able to incorporate smoothness, which requires the value of w_k and some of its partial derivatives to be zero on the boundary of S_k , we are led rather naturally to such uncomplicated shapes as squares and circles. Squares are associated with ℓ_∞ and ℓ_1 norms on 2-space and circles with the ℓ_2 norm. The weight functions for these regions can be easily built up from univariate functions, as follows. Let $\omega(s)$ denote a univariate function with appropriate smoothness. For example

$$\omega(s) = \begin{cases} 1-|s| & |s| \leq 1 \\ 0 & |s| > 1 \end{cases}$$

is continuous, while

$$\omega(s) = \begin{cases} 1-3s^2 + 2|s|^3 & |s| \leq 1 \\ 0 & |s| > 1 \end{cases}$$

has continuous derivatives as well. Similar functions with arbitrarily high smoothness can be generated. Reasonable weight functions for balls of radius r_k centered at (x_k,y_k) are given by:

$$\ell_1 : \quad w_k(x,y) = \omega\left(\frac{(x-x_k) + (y-y_k)}{r_k}\right) \omega\left(\frac{(x-x_k) - (y-y_k)}{r_k}\right)$$

$$\ell_2 : \quad w_k(x,y) = \omega\left(\sqrt{(x-x_k)^2 + (y-y_k)^2}/r_k\right)$$

$$\ell_\infty : \quad w_k(x,y) = \omega\left(\frac{x-x_k}{r_k}\right) \omega\left(\frac{y-y_k}{r_k}\right) .$$

The choice of region will affect the approximation slightly. Which should be used is a matter of personal preference, in the author's opinion. The square regions have the distinction that the weight functions are piecewise polynomials, which results in the function F being a rational function if the U_k are polynomials or rational functions. While this is aesthetically more pleasing than a function involving square roots as is obtained for circles, the author believes the practical difference to be minimal. It may be an advantage to use regions which do not have flat sides. Some calculations have been done both ways with neither one seeming to be significantly better than the other.

We will generally assume that the regions S_k are chosen to contain a set number of nearest neighbors, say $m-1$, of the point (x_k,y_k). This will allow the use of the same type of local interpolating function for each region and will simplify the coding of programs to implement the procedure. The value of m will greatly influence the computational effort required, since the larger m is, the more the support regions S_k will overlap. On the other hand, too small a value for m will not allow the local interpolating function to mimic $f(x,y)$ suitably, and may cause holes in the region on which $F(x,y)$ is defined, particularly if the data is sparsely located in some areas. Hence a balance must be struck between computational efficiency and adequate sampling of $f(x,y)$ along with coverage considerations.

The choice of local interpolating functions U_k is very broad, since essentially the only restrictions are appropriate smoothness and interpolation of the data. The obvious choices are polynomials consisting of m monomials, chosen in some fashion. While this works well for some sets of data, it has not yielded the best approximations overall.

Better results have been obtained by using the optimal approximations previously mentioned [6], [9], [2]. While optimal approximations have not received great attention in applications, they do have very desirable properties and prove to yield quite good results, with one exception, when applied to our test problems in the next section. These functions are splines and thus are better able to adapt to certain data.

When using optimal local interpolating functions it is probably desirable to use weight functions of no more smoothness than the interpolating functions. Thus if the interpolating function is the optimal approximation from the class $B_{[1,1]}$ (see [10] for the definition of this space) one should choose

$$\omega(s) = \begin{cases} 1-|s| & |s| \leq 1 \\ 0 & |s| > 1 \end{cases}$$

since functions in $B_{[1,1]}$ are only continuous.

4.0 Numerical Experiments

It was deemed desirable to test a number of methods on a variety of surfaces for purposes of comparison. Five sets of data were fit using eleven fitting functions, and one set of data was fit using ten of these procedures, the eleventh not being suitable. We will describe the procedures and the data sets and give a table of results with some discussion and observations.

4.1 The Data Sets

Because of the variety of surfaces used by McLain [5], it was decided to use them for our tests also. Unlike McLain, we did not specify uniformly spaced points on $[1,10] \times [1,10]$. Instead, in each of the 81 squares $[i,i+1]\times[j,j+1]$ $i,j = 1,\ldots,9$, a point was generated by a random number generator. These points are shown graphically in Figure 1. In addition, nine points were specified at positions which were judged a priori to be somewhat critical points in defining the surface. Thus a total of 90 points were specified on surfaces S1-S5 on the square $[1,10] \times [1,10]$.

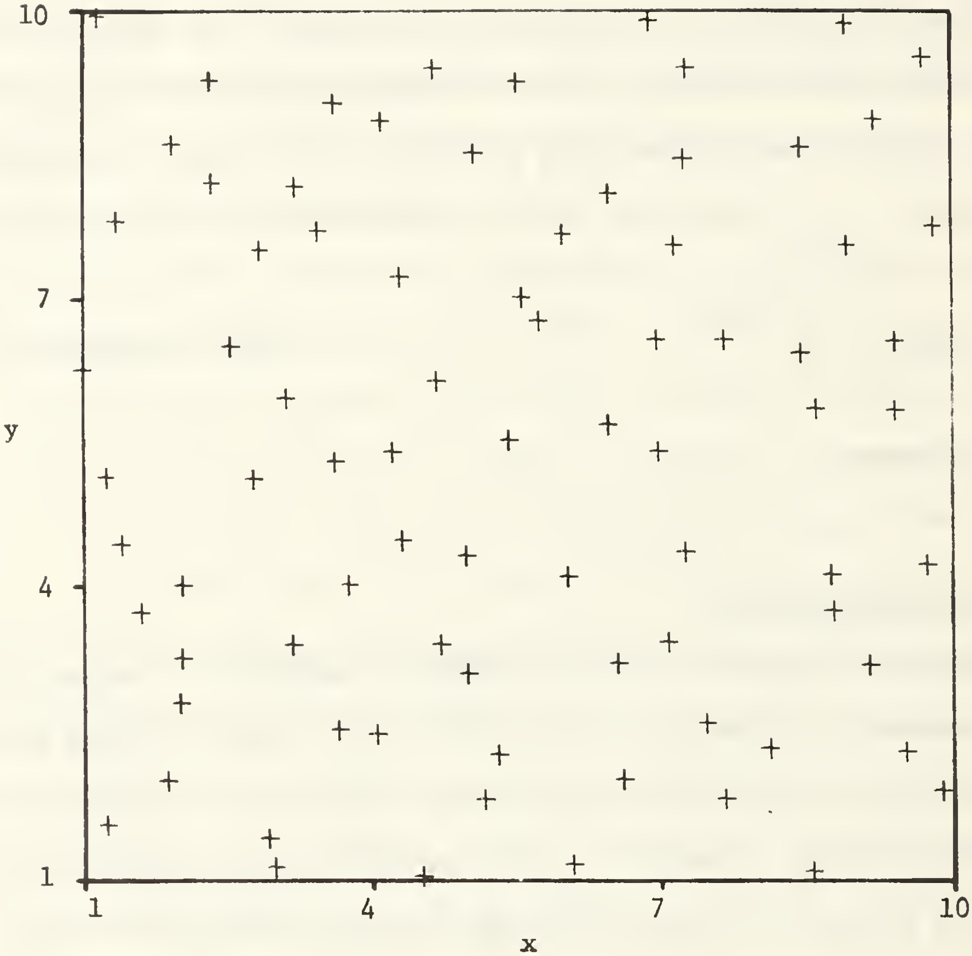

Figure 1: **Interpolation Points**

We point out that McLain was interested in contours for topographical data and he describes the surfaces in topographical terms. We shall do this also as this leads to an easy visualization of the surfaces.

(S1) This surface is a section of a sphere and is described by $f(x,y) = \sqrt{64 - (x - 5.5)^2 - (y - 5.5)^2}$. The additional points specified were (α,β) , $\alpha,\beta = 1, 5.5, 10$.

(S2) This surface is a hill rising steeply from a plain, and is described by $f(x,y) = \exp(-(x-5)^2 - (y-5)^2)$. The additional points specified were (4,5) , (4.5, 4.5) , (4.5, 5.5) , (5,4) , (5,5) , (5,6) , (5.5, 4.5) , (5.5, 5.5), (6,5) .

(S3) This surface is similar to S2, a hill rising less steeply from a plain. It is described by $f(x,y) = \exp\left(-\frac{(x-5)^2 + (y-5)^2}{4}\right)$. The additional points specified were the same as for S2.

(S4) This surface is a long narrow hill rising from a plain, and is described by $f(x,y) = \exp(-(x+y - 11)^2 - (x-y)^2/10)$. The additional points were (3,8), (4.5, 6.5), (4.5, 5.5), (5.5, 4.5), (5.5, 5.5), (5.5, 6.5), (6.5, 4.5), (6.5, 5.5), (8,3) .

(S5) This surface is a plain and a plateau separated by a sharp rise, almost a cliff. It is described by $f(x,y) = \tanh(x + y - 11)$. The additional points were (1,10), (2.5, 8.5), (4,7), (5,5), (5.5, 5.5), (6,6), (7,4), (8.5, 2.5), (10,1) .

(S6) This surface is that given by Ferguson [3] and is not described mathematically. For purposes of better point spacing the y coordinate was multiplied by three. This was almost necessary because a point and its nearest five neighbors would often include five points very nearly on a

straight line when the original data was used. Multiplication of the y-coordina
by a constant is equivalent to using ellipsoidal regions in methods F1 - F5. It
has no effect on methods F6 - F8, and has a very beneficial effect on methods F9 - F10. The data is listed in Table 1.

4.2 The Interpolation Functions

The eleven interpolation functions consist of five different implementations of the method described in Section 2, three global methods for comparison purposes, and three versions of McLain's distance weighted least squares approximation. We will discuss each of these in turn. Several of them involve use of optimal approximations in certain spaces of functions, and references to more information on these topics are [10], [6], [9], and [2].

(F1) This is Maude's method, using U_k as a second degree polynomial interpolating to the point and its five nearest neighbors. This was described in Section 2.1.

(F2) This is a method of Section 2, using the optimal approximation from the class of functions $B_{[1,1]}$ for the local interpolating function for a point and its five nearest neighbors. Six points per region was chosen as being sufficient for coverage, and adequate for function definition, as well as comparable computationally to Maude's method. Circular regions were used with

$$\omega(s) = \begin{cases} 1-|s| & |s| \le 1 \\ 0 & |s| > 1 \end{cases} .$$

Since Barnhill and Nielson [2] have shown that Sard spaces of type B* have a reproducing kernel, one can find the optimal approximation as a linear combination of basis functions (or representers of the point evaluation functionals), one of which is associated with each data point. The basis

x	y	f
2.0	15.0	2.0
2.49	7.647	2.7
2.981	0.291	2.9
3.471	-7.062	3.0
3.961	-14.418	3.0
7.45	12.003	2.0
7.35	6.012	3.0
7.251	0.018	2.5
7.151	-5.973	1.5
7.051	-11.967	2.0
10.901	9.015	1.5
10.751	4.536	1.425
10.602	.06	1.35
10.453	-4.419	1.276
10.304	-8.895	1.2
14.055	10.509	1.0
14.194	6.783	0.8
14.331	3.054	1.2
14.469	0.672	1.6
14.607	-4.398	1.25
15.0	12.0	0.0
15.729	8.067	0.0
16.457	4.134	0.20
17.185	0.198	0.60
17.914	-3.735	1.2

Table 1

function for the point (x_k,y_k) in $B_{[1,1]}$ is

$$K_k(x,y) = [1 + (x-a) - (x-x_k)_+][1 + (y-b) - (y-y_k)_+] ,$$

where $(t)_+^i = \begin{cases} 0 , & t \leq 0 \\ t^i , & t > 0 \end{cases}$ is the truncated power function and

(a,b) is a parameter pair which in this case must satisfy $x_n \geq a$, $y_n \geq b$ for $n = 1,2,\ldots,N$. In general (a,b) need not satisfy this condition, but we have simplified K_k by requiring the condition.

We can note that optimal approximations from $B_{[1,1]}$ are piecewise bilinear functions with knots along lines through the data points and parallel to the axes.

(F3) This is a method of Section 2, similar to function F2, except that U_k is taken to be the optimal approximation from $B_{[2,2]}$, and the weight function used

$$\omega(s) = \begin{cases} 1-3s^2 + 2|s| & |s| \leq 1 \\ 0 & |s| > 1 \end{cases} .$$

Again, the nearest five neighbors in a circular region were used. The basis function associated with the point (x_k,y_k) in this case is

$$K_k(x,y) = g(a,x_k,x)g(b,y_k,y)$$

where $$g(a,x_k,x) = 1 + (x_k - a)(x-a) + \frac{(x_k-a)(x-a)^2}{2!} - \frac{(x-a)^3}{3!} + \frac{(x-x_k)_+^3}{3!} ,$$

where the notation and parameters are as described before.

(F4) This approximation is the same as F3 except that the nine nearest neighbors in a circular region were used to define the local interpolating function. This is to test the effect of including more points in the local approximation.

(F5) This is a method of Section 2, where the U_k is taken to be the optimal approximation from the space of functions $T^{1,1}$ described by Mansfield [6]. Again, a circular region containing a point and its five nearest neighbors was used, with $\omega(s) = \begin{cases} 1-|s| & , |s| \leq 1 \\ 0 & , |s| > 1 \end{cases}$. This space is constrained so as to give an exact approximation if the approximated function is linear. Our version used the basis functions given by Nielson [9] and are of the form

$$K_k(x,y) = \frac{(x-x_k)_+^3}{3!} + \frac{(y-y_k)_+^3}{3!} +$$

$$\left[(x_k - x)_+ - (a - x)_+ - (x_k - a)_+\right]\left[(y_k-y)_+ - (b-y)_+ - (y_k-b)_+\right] ,$$

where a and b are parameters. The optimal approximation is a linear combination of $1,x,y,$ and the K_k.

(F6) This is the global optimal approximation from $B_{[1,1]}$.

(F7) This is the global optimal approximation from $B_{[2,2]}$.

(F8) This is the global optimal approximation from $T^{1,1}$.

(F9) This is McLain's method M10 which he describes as being accurate with a satisfactory amount of computation. The weight attached to a data point (x_k,y_k) is $w(d) = \exp(d^2)/d^2$ where $d^2 = (x-x_k)^2 + (y-y_k)^2$ is the distance squared from the point (x,y) at which the function value is desired. The fitting function is a second degree polynomial.

(F10) This is the same as F9 except that distance is changed. Here $d^2 = [(x-x_k)^2 + (y-y_k)^2]/10$. This is equivalent to shrinking the coordinates by a factor $\sqrt{10}$, and is to test the effect of a different coordinate system.

(F11) This is the same as F9 except that distance is changed. Here $d^2 = 10\left[(x-x_k)^2 + (y-y_k)^2\right]$. This is equivalent to expanding the coordinates by a factor $\sqrt{10}$ and is to test the effect of a different coordinate system.

4.3 Comparisons

For surfaces S1-S5 and all eleven interpolating functions the deviation at the 361 points (x,y), $x,y = 1(.5)10$ was generated. The maximum deviation, the mean deviation, and the root-mean-square deviations are listed in Table 2. The number in parenthesis below each function is the approximate calculation time in seconds for all calculated data points. The calculations were done on the IBM 360/67 at the Naval Postgraduate School. These times are given only as an indication of the number of computations required for the various methods.

Generally we can observe several things from the table:

(i) Maude's method (F1) does not compare favorably with any of the other methods, except when applied to surface S1 .

(ii) Surface S1 seems to be relatively difficult to fit, although Maude's method (F1) and McLain's method (F9) work very well.

(iii) The local weighted optimal approximations (F2-F5) often are more accurate than the corresponding global versions (F6-F8) . A notable exception is the application to surface S1 .

(iv) McLain's method (F9) does very well on all these surfaces and is often the most accurate. The suspicion that the distance weighting function is rather finely tuned for this particular coordinate system is reinforced by poorer performance of the expanded and contracted length versions of it (F10-F11). The amount of computation required is considerably greater for the method than for any other methods considered here, even the global methods.

Surface / Fitting Function	S1	S2	S3	S4	S5
F1 : Maude (34)	.06562 .00390 .00791	1.390 .06882 .2106	1.198 .03964 .1262	3.798 .09371 .3789	.7438 .04440 .10468
F2 : $B_{\lceil 1,1 \rceil}$ local (51)	1.768 .1378 .2456	.1704 .00943 .02661	.1520 .01180 .02429	.3393 .01732 .04202	.4926 .04828 .09124
F3 : $B_{\lceil 2,2 \rceil}$ local (63)	1.284 .1049 .1859	.1390 .00726 .01993	.08185 .00832 .01483	.2701 .01446 .03676	1.111 .03876 .09576
F4 : $B_{\lceil 2,2 \rceil}$ local (143)	.3472 .03970 .06454	.1315 .00727 .01953	.05723 .00593 .01073	.2585 .01758 .03831	1.178 .04932 .11336
F5 : $T^{1,1}$ local (60)	.1068 .01667 .02477	.1740 .00884 .02149	.1143 .01035 .01797	.2398 .01493 .03280	.4656 .03872 .07300
F6 : $B_{\lceil 1,1 \rceil}$ global (128)	.2209 .02724 .05102	.2829 .05706 .08218	.1514 .02234 .03416	.3547 .05453 .08126	.5098 .10322 .14568
F7 : $B_{\lceil 2,2 \rceil}$ global (154)	.09152 .00832 .01465	.4838 .07447 .1086	.09318 .01012 .01719	1.335 .2005 .3127	.8164 .10104 .15686
F8 : $T^{1,1}$ global (138)	.1222 .01355 .02394	.2693 .05135 .07601	.1457 .02001 .03150	.3549 .05106 .07797	.4868 .09632 .13595
F9 : McLain (372)	.05223 .00354 .00827	.1372 .00824 .02090	.04894 .00530 .00873	.1803 .01124 .02546	.4164 .03006 .05960
F10: McLain, $d^2 \leftarrow d^2/10$ (465)	.04343 .00621 .00982	.2158 .02196 .04363	.08728 .02174 .03117	.3894 .02549 .05174	.2738 .05374 .07886
F11: McLain, $d^2 \leftarrow 10d^2$ (87)	5.735 .03785 .3169	1.304 .01222 .07502	.1254 .00982 .02010	.4145 .01539 .03949	4.118 .04584 .2318

Table 2: Maximum, Mean and RMS deviations

(v) While functions in $B_{[1,1]}$ and $T^{1,1}$ are of similar smoothness (only continuous), $T^{1,1}$ gives considerably more flexibility which shows up when fitting surface S1. Results are comparable for surfaces S2-S5.

(vi) Changing the number of neighbors included in a region seems to generally have only a small effect (F3 vs. F4). The one exception, again, is on surface S1 .

For surface S6 we could not compare exact errors since only data points were given. The data is apparently difficult to fit since it involves three relative maxima with a saddle point between two of them and a shallow valley between two of them, the entire surface only having 25 points given on it. In this situation it is not surprising that interpolating surfaces may involve some overshoot. We list the minimum and maximum observed values for functions F1-F10 in Table 3. Again, the flexibility of the space $T^{1,1}$ is noted, and the fine tuning of the distance function for F9 , yielding a good result in F10.

Function	Extreme Values
F1	-6.43, 7.18
F2	0 , 4.65
F3	0 , 5.52
F4	0 , 6.02
F5	0 , 3.00
F6	0 , 3.77
F7	0 , 5.05
F8	0 , 3.05
F9	-26.31, 25.83
F10	0 , 3.30

Table 3

5.0 Conclusion

The main conclusion which can be made is that the method of Section 2 yields good approximations provided the local interpolating functions are reasonably accurate. Six points per region appears to give sufficient function definition and coverage at a reasonable computational load. Any of the optimal approximations seem to be satisfactory as local interpolating functions, and the use of them is strongly recommended over polynomials.

The use of McLain's method (F9) or a variation of it gives very good results if the computational load is not burdensome. Some tuning of the measure of distance may be necessary and is a definite disadvantage of the method.

It appears that on certain surfaces, some methods will do poorly while on others the situation could be reversed. In terms of which of the methods discussed can ultimately be tuned to a given set of data, the author feels McLain's method has a good capability. Unfortunately in case only isolated points are known the procedure for this tuning is not known, and it is probably safer to use a method which adapts naturally to changes in coordinate systems and possible varying sparseness of data points, such as the methods of Section 2, where changing the obvious parameter (the number of neighbors included in a region) seems to have a fairly small influence.

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